MATH 2230 Final exam

Problem 1: (a) $\int_0^{\pi} \frac{d\theta}{(2+\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$

$= \frac{1}{2}\int_0^{2\pi} \frac{\frac{1}{i}e^{-i\theta}\,de^{i\theta}}{\left(2+\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^2} = \frac{1}{2i}\int_0^{2\pi} \frac{4z}{(z^2+4z+1)^2}\,dz$ (If $z=e^{i\theta}$)

$= \frac{2}{i}\int_C \frac{z}{(z+2+\sqrt{3})^2}\left(\frac{1}{z-(-2+\sqrt{3})}\right)^2 dz$ where $C=\{|z|=1\}$

$= \frac{2}{i}(2\pi i)\frac{d}{dz}\left(\frac{z}{(z+2+\sqrt{3})^2}\right)\Bigg|_{z=-2+\sqrt{3}}$ $\quad (|-2+\sqrt{3}|<1)$

$= 4\pi\left(\frac{(z+2+\sqrt{3})^2-2z(z+2+\sqrt{3})}{(z+2+\sqrt{3})^4}\right)\Bigg|_{z=-2+\sqrt{3}} = \frac{2\sqrt{3}\pi}{9}$

(b) Consider the closed contour consisting of upper half circle with radius R and the line segment $z=R$ to $z=-R$, denote it by C_R.

$\cancel{\oint_{C_R}\frac{z^2}{1+z^6}dz}$ If $1+z^6=0 \Rightarrow z=e^{i\pi/6}, e^{i\pi/2}, e^{5i\pi/6}$ ....

we denote these three roots by z_i, $i=1,2,3$.

let $f(z)=\frac{z^2}{1+z^6}$, $\operatorname{Res}_{z=z_i} f(z) = z_i^2 \lim_{z\to z_i}\frac{z-z_i}{1+z^6}$

$= \frac{z_i^2}{6z_i^5}$

$$\int_{C_R} \frac{z^2}{1+z^6}\,dz = 2\pi i\left(\frac{1}{6}\right)\left(e^{-i\pi/2} + e^{-3i\pi/2} + e^{-5i\pi/2}\right) \quad \text{for } R \text{ large enough}$$

$$= \pi/3$$

Denote the upper arc by A_R,

$$\int_{C_R} \frac{z^2}{1+z^6}\,dz = \int_{A_R} \frac{z^2}{1+z^6}\,dz + \int_{-R}^{R} \frac{x^2}{1+x^6}\,dx$$

$$\left|\int_{A_R} \frac{z^2}{1+z^6}\,dz\right| \leq \int_{A_R} \frac{R^2}{R^6-1}\,|dz| \to 0 \quad \text{as } R\to\infty$$

$$\therefore \int_0^{\infty} \frac{x^2}{1+x^6}\,dx = \frac{\pi}{6}$$

(c) Consider C_R in (b),

$$\int_{C_R} \frac{z e^{iz}}{z^2+4}\,dz = \int_{C_R} \frac{z e^{iz}}{z+2i}\cdot\left(\frac{1}{z-2i}\right)dz$$

$$= 2\pi i\left(\frac{2i e^{-2}}{4i}\right) \quad \text{if } R \text{ large enough}$$

$$= \pi i e^{-2}$$

$$\int_{C_R} \frac{z e^{iz}}{z^2+4}\,dz = \int_{A_R} \frac{z e^{iz}}{z^2+4}\,dz + \int_{-R}^{R} \frac{x e^{ix}}{x^2+4}\,dx$$

$$\left|\int_{A_R} \frac{z e^{iz}}{z^2+4}\,dz\right| \leq \frac{\pi R}{R^2-4} \to 0 \quad \text{as } R\to\infty$$

by Jordan Lemma

$$\therefore \int_{-\infty}^{\infty} \frac{x\sin x}{x^2+4}\,dx = \pi e^{-2}$$

ⓐ

Problem 2: Rouché's theorem: If C is a simple closed contour and

① f, g are analytic inside and on C

② $|f| > |g|$ at each point on C,

then f and $f+g$ have the same number of zeros.

Set $f = z^5 + 1$, $g = \alpha z^2$,

On $C = \{|z| = 2\}$, $|f| \geq |z^5| - 1 = 31$

$$|g| \leq |\alpha| \cdot |z^2| \leq 28$$

$\Rightarrow$ $|f| > |g|$

$\Rightarrow$ $\alpha^5 + \alpha z^2 + 1 = 0$ has 5 roots inside C since $z^5 + 1 = 0$ also has 5 roots inside C.

ⓑ We first check that $f = z^5 + (1+2i)z^2 + 1$ has no roots on real and imaginary axis.

If $z = x \in \mathbb{R}$ and $x^5 + (1+2i)x^2 + 1 = 0$

$$\Rightarrow \quad 1 + 2i = \frac{-1 - x^5}{x^2} \in \mathbb{R}$$

which is impossible.

If $z = iy$, $y \in \mathbb{R}$, $iy^5 - (1+2i)y^2 + 1 = 0$

$$1 - y^2 + i(y^5 - 2y^2) = 0$$

$\Rightarrow y = 1$ but $y^5 - 2y^2 = -1 \neq 0$, which is also impossible.

We take a contour C consisting of line segment from 0 to R, the arc of circle of radius R from $\theta=0$ to $\theta=\pi/2$ and the line segment from Ri to 0.

$$\int_C \frac{f'}{f}dz = \int_0^{\pi/2} + \int_{Ri}^{0} + \int_0^R \frac{f'}{f}dz$$

for the first integral, $z=Re^{i\theta}$,

$$\int_0^{\pi/2} \frac{f'}{f}dz = \int_0^{\pi/2} \frac{5z^4+2\alpha z}{z^5+\alpha z^2+1} Re^{i\theta}\cdot i\,d\theta$$

$$= i\int_0^{\pi/2} \frac{5+\frac{2\alpha}{z^4}}{1+\frac{\alpha}{z^3}+\frac{1}{z^5}}d\theta \to \frac{5\pi i}{2} \text{ as } R\to\infty$$

In the line segment from Ri to 0, we let $z=iy$, $y\in\mathbb{R}, y\geq 0$

$$f(z) = iy^5-(1+2i)y^2+1 = 1-y^2+i(y^5-2y^2)$$

$Re(f)=0$ if $y=1$, $Im(f)=0$ if $y=\sqrt[3]{2}, y=0$

In the line segment from 0 to R, we let $z=x$, $x\in\mathbb{R}, x\geq 0$

$$f(z) = x^5+(1+2i)x^2+1 = x^5+x^2+1+2x^2 i$$

$Re(f)>0\ \forall x\geq 0$, $Im(f)=0$ if $x=0$.

Combine these results, we have $\int_{Ri}^{0} + \int_0^R \frac{f'}{f}dz \to \frac{3\pi i}{2}$ as $R\to\infty$

$$\therefore \frac{1}{2\pi i}\int_C \frac{f'}{f}dz = 2.$$

Problem 3: (a) $a_n = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)\,dz}{z^{n+1}}$

where C is any closed contour contained in Ω.

$$|a_n| \le \frac{1}{2\pi}\int_C \left|\frac{f(z)}{z^{n+1}}\right| dz$$

If we take $C =$ circle of radius $R \in (0,\pi)$ centred at $z=0$,

$$|a_n| \le \frac{1}{2\pi}\,\frac{\sin R}{R^{n+1}}\cdot 2\pi R = \frac{\sin R}{R^n} = R^{|n|}\sin R \quad \forall n = -1,-2 \ldots$$

If we take $R \to \pi$, we have $|a_n| = 0$ $\forall n = -1, -2 \ldots$

(b) f can be extended analytically if we define $\hat{f}$ to be its extension and

$$\hat{f} = \begin{cases} f & z \in \{0<|z|<\pi\} \\ \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f}{z}\,dz & z=0 \end{cases}.$$

By (a), we see that $\hat{f}(0) = 0$.

(c) Maximum modulus principle:

If f is analytic and not constant in a domain D, then $|f|$ has no maximum value in D.

We take $R \in (\pi/2, \pi)$, on $\partial B_R(0)$, $|\hat{f}(z)| \le \dfrac{\sin R}{\pi/2}$

By Maximum principle, $|\hat{f}(z)| \le \dfrac{\sin R}{\pi/2}$ $\forall z \in B_R(0)$

If we take $R \to \pi$, then we have $\tilde{f}(z) = 0$ in Ω.

Problem 4(a): It is equivalent to find the 2nd order term in the taylor series of $z^2 \log z$. Let $f(z) = z^2 \log z$

$f' = 2z \log z + z$, $f'' = 3 + 2\log z$

2nd order term in taylor series of $f = \frac{1}{2}\left(3 + 2\log(-1-i)\right)$

$= \frac{1}{2}\left(3 + 2\left(\log\sqrt{2} - \frac{3\pi}{4}\right)\right)$

$= \frac{3}{2} + \log\sqrt{2} - \frac{3\pi}{4}$

This value equals to residue at $z = -1-i$.

(b) For $\frac{2}{e^z - 1}$, since $e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$,

by long division, we have $\frac{2}{e^z - 1} = \frac{2}{z} - 1 + \dots$

For $\frac{a}{\sin z}$, $\operatorname{Res}_{z=0} \frac{a}{\sin z} = \lim_{z \to 0} \frac{az}{\sin z} = a$.

$\therefore$ The residue at 0 equals to $2 + a$.

For g having a removable singularity at 0, we require $\lim_{z \to 0} g \cdot z = 0$ since the Laurent series of g has negative power term up to -1 power. Hence, we require $a = -2$.

Problem 5: (a) We take the contour C consisting of the following four components:

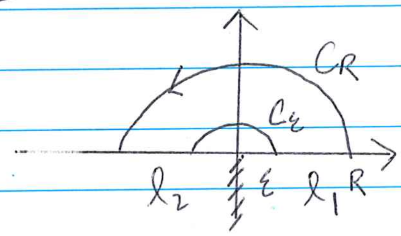
C_R : the arc of upper half circle with radius R

C_ε : the arc of upper half circle with radius ε.

l_1 : the line segment from ε to R.

l_2 : the line segment from $-R$ to $-\varepsilon$.

Also, we define $\log z$ on the branch $-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$.



$$\int_C \frac{\log z}{(z^2+4)}\,dz = \int_C \left(\frac{\log z}{z+2i}\right)\left(\frac{1}{z-2i}\right)dz$$

$$= 2\pi i \left(\frac{\log 2i}{4i}\right)$$

$$= \frac{\pi}{2}\left(\log 2 + \frac{i\pi}{2}\right)$$

$$\left|\int_{C_R} \frac{\log z}{z^2+4}\,dz\right| \leq \pi R \left(\frac{\log R + \pi}{R^2-4}\right) \to 0 \text{ as } R \to \infty.$$

$$\left|\int_{C_\varepsilon} \frac{\log z}{z^2+4}\,dz\right| \leq \pi \varepsilon \left(\frac{|\log \varepsilon| + \pi}{4-\varepsilon^2}\right) \to 0 \text{ as } \varepsilon \to 0.$$

$$\int_{l_2} \frac{\log z}{z^2+4}\,dz = \int_{-R}^{-\varepsilon} \frac{\log x}{x^2+4}\,dx$$

$$= \int_{\varepsilon}^{R} \frac{\log(-x)}{x^2+4}\,dx \quad (\text{Replace } x \text{ by } -x)$$

$$= \int_{\varepsilon}^{R} \frac{\log x + i\pi}{x^2+4}\,dx$$

Since $\int_C \frac{\log z}{z^2+4} dz = \frac{\pi}{2}\left(\log 2 + \frac{i\pi}{2}\right)$, by taking $R \to \infty$,

$\varepsilon \to 0$, we have $2\int_0^\infty \frac{\log x}{x^2+4} dx + i\pi \int_0^\infty \frac{dx}{x^2+4} = \frac{\pi}{2}\left(\log 2 + \frac{i\pi}{2}\right)$

Therefore, $\int_0^\infty \frac{\log x \, dx}{x^2+4} = \frac{\pi \log 2}{4}$

(b) We take the same contour and branch in (a),

Pick, ε small and R large enough,

$$\int_C \frac{z^{1/3}}{(z^2+1)^2} dz = \int_C \frac{z^{1/3}}{(z+i)^2(z-i)^2} dz$$

$$= 2\pi i \left.\frac{d}{dz}\left(\frac{z^{1/3}}{(z+i)^2}\right)\right|_{z=i} \quad \left(z^{1/3} \text{ is analytic in the interior of } C\right)$$

$$= 2\pi i \left.\left(\frac{(z+i)^2 \, {}^1\!/\!_3\, z^{-2/3} - z^{1/3}\, 2(z+i)}{(z+i)^4}\right)\right|_{z=i}$$

$$= \pi/2 \left(\sqrt{3}/3 + i/3\right)$$

$$\left|\int_{C_R} \frac{z^{1/3}}{(z^2+1)^2} dz\right| \leq \int_{C_R} \frac{\left|e^{1/3 \log z}\right|}{(R^2-1)^2} dz = \int_{C_R} \frac{\left|e^{1/3(\log R + i\theta)}\right|}{(R^2-1)^2} dz$$

$$= \pi R \left(\frac{R^{1/3}}{(R^2-1)^2}\right) \to 0 \text{ as } R \to \infty$$

$$\left|\int_{C_\varepsilon} \frac{z^{1/3}}{(z^2+1)^2} dz\right| \leq \int_{C_\varepsilon} \frac{\left|e^{1/3(\log \varepsilon + i\theta)}\right|}{(1-\varepsilon^2)^2} dz = \pi\varepsilon \left(\frac{e^{1/3 \log \varepsilon}}{(1-\varepsilon^2)^2}\right)$$

$\to 0$ as $\varepsilon \to 0$.

Since $\int_C \frac{z^{1/3}}{(z^2+1)^2} dz = \frac{\pi}{2}\left(\frac{\sqrt{3}}{3} + \frac{i}{3}\right)$, by taking $R \to \infty$, $\varepsilon \to 0$

$$\int_0^\infty \frac{x^{1/3}}{(x^2+1)^2} dx + \int_{-\infty}^0 \frac{z^{1/3}}{(z^2+1)^2} dz = \frac{\pi}{2}\left(\frac{\sqrt{3}}{3} + \frac{i}{3}\right)$$

$$\int_0^\infty \frac{x^{1/3}}{(x^2+1)^2} dx + \int_0^\infty \frac{e^{\frac{1}{3}\log(-x)}}{(x^2+1)^2} dx = \frac{\pi}{2}\left(\frac{\sqrt{3}}{3} + \frac{i}{3}\right)$$

$$\int_0^\infty \frac{x^{1/3}}{(x^2+1)^2} dx + \int_0^\infty \frac{e^{\frac{1}{3}(\log x + \log(-1))}}{(x^2+1)^2} = \frac{\pi}{2}\left(\frac{\sqrt{3}}{3} + \frac{i}{3}\right)$$

$$\left(\frac{3}{2} + \frac{\sqrt{3}}{2} i\right) \int_0^\infty \frac{x^{1/3}}{(x^2+1)^2} dx = \frac{\pi}{2}\left(\frac{\sqrt{3}}{3} + \frac{i}{3}\right)$$

$$\Rightarrow \int_0^\infty \frac{x^{1/3}}{(x^2+1)^2} dx = \frac{\sqrt{3}}{9}\pi$$